3.3: Homogeneous Equations with Constant Coefficients

We restrict our attention to equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{1}$$

and define the **characteristic** (or **auxiliary**) equation by

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0. \tag{2}$$

Theorem 1. (Distinct Real Roots)
If the roots $r_1, r_2, \ldots, r_n$ of the characteristic equation (2) are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

is a general solution to (1).

Example 1. Solve the initial value problem

$$y^{(3)} + 3y'' - 10y' = 0, \quad y(0) = 7, y'(0) = 0, y''(0) = 70.$$

$r^3 + 3r^2 - 10r = 0$

$r(r^2 + 3r - 10) = 0$

$r(r+5)(r-2) = 0$

$r_1 = 0, r_2 = -5, r_3 = 2$

$y = c_1 + c_2 e^{-5x} + c_3 e^{2x}, \quad 7 = c_1 + c_2 + c_3$

$y' = -5c_2 e^{-5x} + 2c_3 e^{2x}, \quad 0 = -5c_2 + 2c_3$

$y'' = 25c_2 e^{-5x} + 4c_3 e^{2x}, \quad 70 = 25c_2 + 4c_3$

$c_1 = 0, c_2 = 2, c_3 = 5$

$y = 2e^{-5x} + 5e^{2x}.$

Theorem 2. (Repeated Roots)
If the characteristic equation (2) has a repeated real root r of multiplicity k, then the part of a general solution of the differential equation (1) corresponding to r is of the form

$$(c_1 + c_2 x + \cdots + c_{k-1} x^{k-2} + c^k x^{k-1}) e^{rx}.$$

Example 2. Find a general solution to the equation

$$9y^{(5)} - 6y^{(4)} + y^{(3)} = 0.$$

$9r^5 - 6r^4 + r^3 = 0$

$r^3(9r^2 - 6r + 1) = 0$

$r^3(3r-1)^2 = 0$

$r_1 = r_2 = r_3 = 0, r_4 = r_5 = \frac{1}{3}$

$y(x) = (c_1 + c_2 x + c_3 x^2) e^{0x} + (c_4 + c_5 x) e^{x/3}$

$= c_1 + c_2 x + c_3 x^2 + c_4 e^{x/3} + c_5 x e^{x/3}$

Next we wish to consider what happens if we do not have all real roots. In order to do this, we use Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{3}$$

to give us a hint. Therefore

$$e^{(a+ib)x} = e^{ax}(\cos bx + i\sin bx).$$

Theorem 3. (Complex Roots)
If the characteristic equation (2) has a complex root $r = a + ib$, then the part of a general solution of the differential equation (1) corresponding to r is of the form

$$e^{ax}(c_1\cos bx + c_2\sin bx).$$

Exercise 1. Show that $y(x) = c_1\cos bx + c_2\sin bx$ is a solution to

$$y'' + b^2 y = 0.$$

$$\left.\begin{aligned} y &= c_1\cos bx + c_2\sin bx \\ y' &= -bc_1\sin bx + bc_2\cos bx \\ y'' &= -b^2c_1\cos bx - b^2c_2\sin bx \end{aligned}\right\} \checkmark$$

Example 3. Find the particular solution to the initial value problem

$$y'' - 4y' + 5y = 0, \quad y(0) = 1, y'(0) = 5.$$

$$r^2 - 4r + 5 = 0$$

$$\frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i \Rightarrow r_1 = 2 - i,\ r_2 = 2 + i$$

Thus $y(x) = e^{2x}(c_1\cos x + c_2\sin x)$, $y(0) = 1 = c_1$

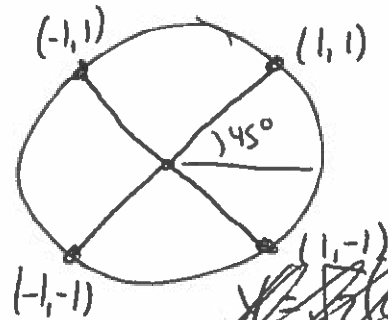
$$y'(x) = 2e^{2x}(\cos x + c_2\sin x) + e^{2x}(-\sin x + c_2\cos x)$$

$$y'(0) = 5 = 2 + c_2 \Rightarrow c_2 = 3$$

$$y(x) = e^{2x}(\cos x + 3\sin x)$$

Exercise 2. Find the general solution to

$$y^{(4)} + 4y = 0.$$



$$r^4 + 4 = 0$$

$$(r^2+2i)(r^2-2i) = 0$$

$$(r+\sqrt{2i})(r-\sqrt{2i})(r+\sqrt{-2i})(r-\sqrt{-2i}) = 0$$

$$r_1 = \sqrt{2}\, e^{i\pi/4},\ r_2 = \sqrt{2}\, e^{i3\pi/4},\ r_3 = \sqrt{2}\, e^{-i\pi/4},\ r_4 = \sqrt{2}\, e^{-3i\pi/4}$$

$$y = e^{x}(c_1\cos x + c_2 \sin x) + e^{-x}(c_3 \cos x + c_4 \sin x)$$

Theorem 4. (Repeated Complex Roots)
If the characteristic equation (2) has a repeated complex root $r = a + ib$ of multiplicity k, then the part of a general solution of the differential equation (1) corresponding to r is of the form

$$\sum_{p=0}^{k-1} x^p e^{ax}(c_p \cos bx + c_p \sin bx).$$

Example 4. Find the general solution to the differential equation whose characteristic equation has roots $3, -5, 0, 0, 0, 0, -5, 2 \pm 3i$ and $2 \pm 3i$.

Roots	Solution
0 has multiplicity 4	$c_1 + c_2 x + c_3 x^2 + c_4 x^3$
3 has multiplicity 1	$c_5 e^{3x}$
-5 has multiplicity 2	$(c_6 + c_7 x)e^{-5x}$
$2 \pm 3i$ has multiplicity 2	$e^{2x}(c_8 \cos 3x + c_9 \sin 3x) + xe^{2x}(c_{10}\cos 3x + c_{11}\sin 3x)$

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{3x} + (c_6 + c_7 x)e^{-5x} + e^{2x}(c_8 \cos 3x + c_9 \sin 3x)$$
$$+ xe^{2x}(c_{10}\cos 3x + c_{11}\sin 3x)$$

Exercise 3. Find the general solution to the differential equation

$$y^{(3)} + y' - 10y = 0.$$

$$r^3 + r - 10 = 0$$

Only possible Roots are $\pm 1, \pm 2, \pm 5, \pm 10$

$$(r-2)(r^2+2r+5) = 0$$

$$(r-2)[(r+1)^2+4] = 0$$

$$r_1 = 2, \quad r_2 = -1 \pm 2i$$

$$y(x) = c_1 e^{2x} + e^{-x}(c_2 \cos 2x + c_3 \sin 2x)$$

Homework. 1-31 (odd)